LECTURE 25 ROLLE'S THEOREM AND THE MEAN VALUE THEOREM

Theorem. *(Rolle's Theorem) Suppose f is a continuous function on $[a, b]$ and differentiable on (a, b) (at every interior point of $[a, b]$). If $f(a) = f(b)$, then there exists at least $c \in (a, b)$ such that $f'(c) = 0$.*

Sketch of proof: To see why this is true, we consider what kind of points may give you extreme values.
at interior points such that $f' = 0$.
at interior points such that f' is undefined.
endpoints.
Now, since f is hypothesized to be differentiable, the second case is ruled out.

For the first case, if extreme values occurs on the interior, then they are local extrema and thus have $f' = 0$, automatically yielding the claim.

For the third case, both the absolute max and absolute min occur at the endpoints. Since we hypothesize that $f(a) = f(b)$ which means both endpoints are absolute max AND min, implying that the function must be constant on the interval. Constant functions have zero slope everywhere – this gives the conclusion of the Rolle's theorem.

Remark. Rolle's theorem is often used in a proof technique called "proof by contradiction". The procedure is as follows. Suppose you want to show that statement A is true. You always keep track of a pool of truths already given or obtained. First you assume that statement A is false. This could possibly imply that one of the statements in the pool of truth is false. Now, as a statement cannot be both true and false at the same time, this can only mean that your initial assumption that statement A is false is wrong. This implies, in turn, that A must be true.

Example. Show that there is exactly one real solution to the equation $x^3 + 3x + 1 = 0$.

Solution. This problem may seem unrelated to Rolle's theorem. But it does appear as a proper application of proof by contradiction.

The first thing we always do is to define a function $f(x) = x^3 + 3x + 1$. We are interested in knowing how many x's makes $f(x) = 0$, namely, how many times the graph of f crosses the x-axis.

We know by the Intermediate Value Theorem that if a continuous function has a sign change at the endpoints of a closed interval, then it must have crossed the x-axis at least once. Indeed, $f(x)$ here is a continuous function (a polynomial), and we find that

$$f(0) = 1 > 0$$

and

$$f(-1) = -1 - 3 + 1 = -3 < 0.$$

This implies that f has **at least one** zeros on the interval $(-1, 0)$. This further implies an almost trivial statement that f has **at least one** zeros on $(-\infty, \infty)$ since it is a larger interval containing $(-1, 0)$. Let's record this true statement as

A: f has at least one zeros on the interval $(-\infty, \infty)$.

How do you show that f has **exactly** one root when you know it has at least one? It suffices to show at most one, that is, we need to prove the statement that there is at most one zero on the interval $(-\infty, \infty)$. Let's call this

B: there is at most one zero on the whole domain $(-\infty, \infty)$.

Here comes a proof by contradiction using Rolle's theorem. Suppose the statement B we want to prove is false, that is, "not B is true". We can simply suppose that there are two roots on the interval $(-\infty, \infty)$. Keep in mind that the theorem is an absolute true statement – so it cannot be false.

Having two roots means there are two numbers $a, b \in (-\infty, \infty)$ such that $f(a) = f(b) = 0$. This implies, by Rolle's theorem, that there exists one number $c \in (a, b)$ such that $f'(c) = 0$. However, $f'(x) = 3x^2 + 3 > 0$

for all x, and thus in particular, $f'(c) > 0$. This is contradicting the previous conclusion $f'(c) = 0$ from Rolle's theorem (which is always true given the hypothesis). Reductio ad absurdum! This means, our original assumption that that "not B is true" is false, which means B is true.

Combining true statements A and B, we arrive at the desired conclusion, i.e. there is exactly one real solution to the equation.

Remark. When the problem does not specify where you should locate the root, you ought to first find the interval by means of the Intermediate Value Theorem. This narrows down your search area.

Example. Show that the function $f(x) = x^3 + \frac{4}{x^2} + 7$ has exactly one zero in $(-\infty, 0)$.

Solution. Two steps.

(1) Show at least one root using Intermediate Value Theorem.

First, note that f is a continuous function on the domain as it is a linear combination of polynomials and rational functions without hitting an asymptote (it does when $x = 0$ but 0 isn't included in the domain). Note that our given interval is not closed. But it is okay since our function is continuous on the domain, so we are allowed to take limit and extend our search to the closed interval $[-\infty, 0]$ where the endpoints are understood in the limit sense. All we need to confirm is that this function crosses the x-axis.

$$\lim_{x \to 0^-} f(x) = \infty > 0$$

and

$$\lim_{x \to -\infty} f(x) = -\infty < 0.$$

Therefore, there exists at least one root on the interval $(-\infty, 0)$.

(2) Show at most one root via a proof by contradiction using Rolle's Theorem.

We suppose that there are two roots $x = a$ and $x = b$ on the interval $(-\infty, 0)$. Rolle's theorem tells us then that there exists $c \in (a, b)$ such that $f'(c) = 0$. However, $f'(x) = 3x^2 - \frac{8}{x^3}$, we find that the points that make $f'(x) = 0$ satisfy

$$3x^2 - \frac{8}{x^3} = 0 \implies \frac{3x^5 - 8}{x^3} = 0 \implies x = \left(\frac{8}{3}\right)^{\frac{1}{5}} \notin (-\infty, 0).$$

Therefore, $f'(x)$ is never equal to 0 on $(-\infty, 0)$, contradicting the conclusion of the theorem $f'(c) = 0$ for $c \in (a, b) \subset (-\infty, 0)$. Hence, our assumption that there are two roots on $(-\infty, 0)$ is wrong. There can at most be one root.

Combining statement 1 and 2, we conclude that there exists exactly one root on $(-\infty, 0)$.